

OBSTACLE PROBLEMS WITH MEASURE DATA

Paolo DALL'AGLIO
Chiara LEONE

Abstract

We give a definition for Obstacle Problems with measure data and general obstacles. For such problems we prove existence and uniqueness of solutions and consistency with the classical theory of Variational Inequalities. Continuous dependence with respect to data is discussed.

1. Introduction

In this paper we consider the obstacle problem with measure data for a linear differential operator \mathcal{A} , for which we prove existence and uniqueness of solutions together with some stability results.

Consider first the objects that won't change throughout the work.

Let Ω be a regular bounded open subset of \mathbb{R}^N (for the notion of regularity see Definition 1.1).

Let $\mathcal{A}(u) = -\operatorname{div}(A(x)\nabla u)$ be a linear elliptic operator with coefficients in $L^\infty(\Omega)$, that is $A(x) = (a_{ij}(x))$ is an $N \times N$ matrix such that

$$a_{ij} \in L^\infty(\Omega) \quad \text{and} \quad \sum a_{ij}(x)\xi_i\xi_j \geq \gamma|\xi|^2, \quad \forall \xi \in \mathbb{R}^N, \quad \text{a.e. in } \Omega.$$

We want to consider the obstacle problem also in the case of thin obstacles, so we will need the techniques of capacity theory. For this theory we refer, for instance, to [12].

We recall very briefly that, given a set $E \subseteq \Omega$, its capacity with respect to Ω is given by

$$\operatorname{cap}(E, \Omega) = \inf\{\|z\|_{H_0^1(\Omega)}^2 : z \in H_0^1(\Omega), z \geq 1 \text{ a.e. in a neighbourhood of } E\}.$$

When the ambient set Ω is clear from the context we will write $\operatorname{cap}(E)$.

A property holds quasi everywhere (abbreviated as q.e.) when it holds up to sets of capacity zero.

A function $v : \Omega \rightarrow \overline{\mathbb{R}}$ is quasi continuous (resp. quasi upper semicontinuous) if, for every $\varepsilon > 0$ there exists a set E such that $\operatorname{cap}(E) < \varepsilon$ and $v|_{\Omega \setminus E}$ is continuous (resp. upper semicontinuous) in $\Omega \setminus E$.

We recall also that if u and v are quasi continuous functions and $u \leq v$ a.e. in Ω then also $u \leq v$ q.e. in Ω .

A function $u \in H_0^1(\Omega)$ always has a quasi continuous representative, that is there exists a quasi continuous function \tilde{u} which equals u a.e. in Ω . We shall always identify u with its quasi continuous representative. We also have that if $u_n \rightarrow u$ strongly in $H_0^1(\Omega)$ then there exists a subsequence which converges quasi everywhere.

Consider the function $\psi : \Omega \rightarrow \overline{\mathbb{R}}$, and let the convex set be

$$K_\psi(\Omega) := \{z \text{ quasi continuous} : z \geq \psi \text{ q.e. in } \Omega\}.$$

Without loss of generality we may always assume that ψ is quasi upper semicontinuous thanks to Proposition 1.5 in [8].

In their natural setting, obstacle problems are part of the theory of Variational Inequalities (for which we refer to well known books such as [14] and [18]).

For any datum $F \in H^{-1}(\Omega)$ the Variational Inequality with obstacle ψ

$$\begin{cases} \langle \mathcal{A}u, v - u \rangle \geq \langle F, v - u \rangle & \forall v \in K_\psi(\Omega) \cap H_0^1(\Omega) \\ u \in K_\psi(\Omega) \cap H_0^1(\Omega) \end{cases} \quad (1.1)$$

(which, for simplicity, will be indicated by $VI(F, \psi)$), has a unique solution, whenever the set $K_\psi(\Omega) \cap H_0^1(\Omega)$ is nonempty, that is ensured by the condition

$$\exists z \in H_0^1(\Omega) : z \geq \psi \text{ q.e. in } \Omega. \quad (1.2)$$

In this frame, among all classical results, we recall that the solution of $VI(F, \psi)$ is also characterized as the smallest function $u \in H_0^1(\Omega)$ such that

$$\begin{cases} \mathcal{A}u - F \geq 0 \text{ in } \mathcal{D}'(\Omega) \\ u \geq \psi \text{ q.e. in } \Omega. \end{cases} \quad (1.3)$$

Then $\lambda := \mathcal{A}u - F$ is a positive measure, and will be called the obstacle reaction associated with u .

In order to study the case of right-hand side measure, we recall that, already in the case of equations, the term $\langle \mu, u \rangle$ has not always meaning when μ is a measure and $u \in W_0^{1,q}(\Omega)$, $q < N$. Hence the classical formulation of the variational inequality fails.

Also the use of the characterization (1.3) to define the Obstacle Problem with measure data is not possible because another problem arises: a famous example by J. Serrin (see [16] and, for more details, [15]) shows that the homogeneous equation

$$\begin{cases} \mathcal{A}u = 0 \text{ in } \mathcal{D}'(\Omega) \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

has a nontrivial solution v which does not belong to $H_0^1(\Omega)$. Here \mathcal{A} is a particular linear elliptic operator with discontinuous coefficients. The function $u = 0$ is obviously the unique solution in $H_0^1(\Omega)$.

So (1.3) in general does not determine the solution of the Obstacle Problem: indeed, with such \mathcal{A} , if we choose $\psi \equiv -\infty$, and if u were the minimal supersolution, then we would have $u \leq u + tv$ a.e. in Ω for any t in \mathbb{R} , which is a contradiction.

G. Stampacchia overcame this difficulty for equations, using a wider class of test functions, and gave in [17] the following definition, which uses regularity and duality arguments.

For this theory we need to assume that the boundary $\partial\Omega$ has the following property, which is satisfied in particular when $\partial\Omega$ is Lipschitz.

Definition 1.1. We say that a bounded open subset of \mathbb{R}^N is regular if there exists a constant $\alpha \in (0, 1)$ such that, for every $x_0 \in \partial\Omega$ and for all ρ small, we have

$$|B_\rho(x_0) \setminus \Omega| > \alpha |B_\rho(x_0)|.$$

The theory by Stampacchia actually works under slightly weaker but more complicated assumptions as said in [17] (see Definition 6.2).

Let $\mu \in \mathcal{M}_b(\Omega)$, where $\mathcal{M}_b(\Omega)$ is the space of bounded Radon measures, viewed as the dual of the Banach space $C_0(\Omega)$ of continuous functions that are zero on the boundary.

Definition 1.2. A function $u_\mu \in L^1(\Omega)$ is a solution in the sense of Stampacchia (also called solution by duality) of the equation

$$\begin{cases} \mathcal{A}u_\mu = \mu & \text{in } \Omega \\ u_\mu = 0 & \text{on } \partial\Omega, \end{cases}$$

if

$$\int_{\Omega} u_\mu g \, dx = \int_{\Omega} u_g^* \, d\mu, \quad \forall g \in L^\infty(\Omega),$$

where u_g^* is the solution of

$$\begin{cases} \mathcal{A}^*u_g^* = g & \text{in } H^{-1}(\Omega) \\ u_g^* \in H_0^1(\Omega) \end{cases}$$

and \mathcal{A}^* is the adjoint of \mathcal{A} .

Stampacchia proved that a solution u_μ exists and is unique and belongs to $W_0^{1,q}(\Omega)$, where q is any exponent satisfying $1 < q < \frac{N}{N-1}$. He proved also that, if the datum

μ is more regular, namely in $\mathcal{M}_b(\Omega) \cap H^{-1}(\Omega)$, then the solution coincides with the variational one. It is also known that $T_k(u_\mu)$ belongs to $H_0^1(\Omega)$, where T_k denotes the usual truncation function defined by

$$T_k(s) := (-k) \vee (s \wedge k).$$

From this it follows that every Stampacchia solution u_μ has a quasi continuous representative; in the rest of the paper we shall always identify u_μ with its quasi continuous representative.

Moreover when the data converge $*$ -weakly in $\mathcal{M}_b(\Omega)$, also the solutions converge strongly in $W_0^{1,q}(\Omega)$ and their truncates weakly in $H_0^1(\Omega)$.

We will use the following notation: u_μ denotes the solution of the equation

$$\begin{cases} \mathcal{A}u_\mu = \mu & \text{in } \Omega \\ u_\mu = 0 & \text{on } \partial\Omega, \end{cases}$$

when μ is either a measure in $\mathcal{M}_b(\Omega)$ or an element of $H^{-1}(\Omega)$. In the first case we refer to the definition by G. Stampacchia, in the latter to the usual variational one.

Following these ideas we give a formulation for Obstacle Problems which involves this type of solutions.

Definition 1.3. We say that the function $u \in K_\psi(\Omega) \cap W_0^{1,q}(\Omega)$, $1 < q < \frac{N}{N-1}$ is a solution of the Obstacle Problem with datum μ and obstacle ψ if

1. there exists a positive bounded measure $\lambda \in \mathcal{M}_b^+(\Omega)$ such that

$$u = u_\mu + u_\lambda;$$

2. for any $\nu \in \mathcal{M}_b^+(\Omega)$, such that $v = u_\mu + u_\nu$ belongs to $K_\psi(\Omega)$, we have

$$u \leq v \quad \text{q.e. in } \Omega.$$

Also here the positive measure λ , which is uniquely defined, will be called the obstacle reaction relative to u . This problem will be shortly indicated by $OP(\mu, \psi)$.

To show that for any datum μ there exists one and only one solution, we introduce the set

$$\mathcal{F}_\psi(\mu) := \left\{ v \in K_\psi(\Omega) \cap W_0^{1,q}(\Omega) : \exists \nu \in \mathcal{M}_b^+(\Omega) \text{ s.t. } v = u_\mu + u_\nu \right\}.$$

We will prove that $\mathcal{F}_\psi(\mu)$ has a minimum element, that is a function $u \in \mathcal{F}_\psi(\mu)$ such that $u \leq v$ q.e. in Ω for any other function $v \in \mathcal{F}_\psi(\mu)$. This is clearly the solution of the Obstacle Problem according to the Definition 1.3. If this solution exists it is obviously unique.

Hypothesis (1.2) does not ensure that $\mathcal{F}_\psi(\mu)$ be nonempty. The minimal hypothesis, instead of (1.2), will be

$$\exists \rho \in \mathcal{M}_b(\Omega) : u_\rho \geq \psi \text{ q.e. in } \Omega; \quad (1.4)$$

so the set $\mathcal{F}_\psi(\mu)$ is nonempty for every $\mu \in \mathcal{M}_b(\Omega)$, because it contains the function $u_{\mu^+} + u_\rho$.

The proof of existence will be first worked out for the case of a negative obstacle (Section 2): this is based on an approximation technique. The obstacle reactions associated with the solutions for regular data are shown to satisfy an estimate on the masses, which allows to pass to the limit and obtain the solution in the general case. Then the proof is easily extended to the case of general obstacle (Section 3)

In Section 4 we will give some stability results, and in Section 5 we will show that the classical solution to the Obstacle Problem (equation (1.1)) coincides with the new one (Definition 1.3) when both make sense.

Moreover we will show that this solution coincides with the one given in the wider setting of nonlinear monotone operators, but in the case of datum in $L^1(\Omega)$, by L. Boccardo and T. Gallouët in [3], and by L. Boccardo and G.R. Cirmi in [1] and [2].

Section 6 provides a characterization of the solution in terms of approximating sequences of solutions of Variational Inequalities, and in Section 7 we study some properties of the solution of the Obstacle Problem for the class of Radon measures, that do not charge the sets of zero 2-Capacity.

2. Nonpositive obstacles

Throughout this chapter we assume the obstacle to be nonpositive. In this frame both hypotheses (1.4) and (1.2) are trivially satisfied.

We begin with a preparatory result which will be proved in two steps.

Lemma 2.1. *Let $\psi \leq 0$ and let $\mu \in \mathcal{M}_b(\Omega) \cap H^{-1}(\Omega)$ such that μ^+ and μ^- belong to $H^{-1}(\Omega)$. Let u be the solution of $VI(\mu, \psi)$ and λ the obstacle reaction associated with u . Then*

$$\|\lambda\|_{\mathcal{M}_b(\Omega)} \leq \|\mu^-\|_{\mathcal{M}_b(\Omega)}.$$

Proof. Observe that the function u_{μ^+} is positive and hence greater than or equal to ψ , belongs to $H_0^1(\Omega)$, and

$$\mathcal{A}u_{\mu^+} - \mu \geq 0 \quad \text{in } \mathcal{D}'(\Omega).$$

By (1.3) we have

$$u = u_{\mu} + u_{\lambda} \leq u_{\mu^+} \quad \text{q.e. in } \Omega,$$

and, by linearity,

$$u_{\lambda} \leq u_{\mu^-} \quad \text{q.e. in } \Omega. \tag{2.1}$$

We will prove that this implies

$$\lambda(\Omega) \leq \mu^-(\Omega) \tag{2.2}$$

which is equivalent to the thesis.

To prove (2.2) we note that, thanks to (2.1)

$$\int_{\Omega} w d\mu^- = \langle \mathcal{A}^*w, u_{\mu^-} \rangle \geq \langle \mathcal{A}^*w, u_{\lambda} \rangle = \int_{\Omega} w d\lambda, \tag{2.3}$$

for every $w \in H_0^1(\Omega)$, such that $\mathcal{A}^*w \geq 0$ in $\mathcal{D}'(\Omega)$.

It is now easy to find a sequence $\{w_n\}$ in $H_0^1(\Omega)$ such that $w_n \nearrow 1$ and $\mathcal{A}^*w_n \geq 0$ in $\mathcal{D}'(\Omega)$. For instance, one can choose as w_n the \mathcal{A}^* -capacitary potential (see [12], chapter 9) of J_n , where J_n is an invading family of compact subsets of Ω .

Passing to the limit in (2.3), as $n \rightarrow \infty$, we obtain (2.2). \square

Theorem 2.2. *Let $\psi \leq 0$ and $\mu \in \mathcal{M}_b(\Omega) \cap H^{-1}(\Omega)$. Let u be the solution of $VI(\mu, \psi)$ and let λ be the obstacle reaction relative to u . Then*

$$\|\lambda\|_{\mathcal{M}_b(\Omega)} \leq \|\mu^-\|_{\mathcal{M}_b(\Omega)}. \tag{2.4}$$

Proof. Thanks to Lemma 3.3 in [10] there exists a sequence of smooth functions f_n such that

$$\|f_n - \mu\|_{H^{-1}(\Omega)} \leq \frac{1}{n} \quad \text{and} \quad \|f_n\|_{L^1(\Omega)} \leq \|\mu\|_{\mathcal{M}_b(\Omega)}.$$

Thanks to the next Lemma, the sequence f_n satisfies

$$f_n^\pm \rightharpoonup \mu^\pm \quad \text{*}-\text{weakly in } \mathcal{M}_b(\Omega) \text{ and } \|f_n^\pm\|_{L^1(\Omega)} \rightarrow \|\mu^\pm\|_{\mathcal{M}_b(\Omega)}.$$

Let u_n and u be the solutions of $VI(f_n, \psi)$ and $VI(\mu, \psi)$, respectively. We know from the general theory (see, for instance, [14]) that $u_n \rightarrow u$ in $H_0^1(\Omega)$. So the measures λ_n and λ associated with u_n and u , respectively, satisfy

$$\lambda_n \rightarrow \lambda \text{ in } H^{-1}(\Omega),$$

$$\|\lambda_n\|_{\mathcal{M}_b(\Omega)} \leq \|f_n^-\|_{L^1(\Omega)}.$$

So $\lambda_n \rightharpoonup \lambda$ in $\mathcal{M}_b(\Omega)$, and we get the inequality (2.4). \square

The following lemma is quite simple, but is proved here for the sake of completeness.

Lemma 2.3. *Let μ_n and μ be measures in $\mathcal{M}_b(\Omega)$ such that*

$$\mu_n \rightharpoonup \mu \quad \text{*}-\text{weakly in } \mathcal{M}_b(\Omega) \text{ and } \|\mu_n\|_{\mathcal{M}_b(\Omega)} \rightarrow \|\mu\|_{\mathcal{M}_b(\Omega)}$$

then

$$\mu_n^+ \rightharpoonup \mu^+ \text{ and } \mu_n^- \rightharpoonup \mu^- \quad \text{*}-\text{weakly in } \mathcal{M}_b(\Omega),$$

and

$$\|\mu_n^+\|_{\mathcal{M}_b(\Omega)} \rightarrow \|\mu^+\|_{\mathcal{M}_b(\Omega)} \text{ and } \|\mu_n^-\|_{\mathcal{M}_b(\Omega)} \rightarrow \|\mu^-\|_{\mathcal{M}_b(\Omega)}. \quad (2.5)$$

Proof. Observe that

$$\|\mu_n^\pm\|_{\mathcal{M}_b(\Omega)} \leq \|\mu_n\|_{\mathcal{M}_b(\Omega)}.$$

so, up to a subsequence,

$$\mu_n^+ \rightharpoonup \alpha \text{ and } \mu_n^- \rightharpoonup \beta \quad \text{*}-\text{weakly in } \mathcal{M}_b(\Omega);$$

where $\alpha - \beta = \mu$. Hence, we can compute

$$\begin{aligned} \|\alpha\|_{\mathcal{M}_b(\Omega)} + \|\beta\|_{\mathcal{M}_b(\Omega)} &\leq \liminf \|\mu_n^+\|_{\mathcal{M}_b(\Omega)} + \liminf \|\mu_n^-\|_{\mathcal{M}_b(\Omega)} \\ &\leq \liminf \|\mu_n\|_{\mathcal{M}_b(\Omega)} = \|\mu\|_{\mathcal{M}_b(\Omega)}; \end{aligned}$$

from which we easily deduce that $\alpha = \mu^+$, $\beta = \mu^-$. Therefore the whole sequences μ_n^+ and μ_n^- converge to μ^+ and μ^- respectively. Moreover, as

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \|\mu_n^+\|_{\mathcal{M}_b(\Omega)} + \liminf_{n \rightarrow +\infty} \|\mu_n^-\|_{\mathcal{M}_b(\Omega)} \\ \leq \lim_{n \rightarrow +\infty} \|\mu_n\|_{\mathcal{M}_b(\Omega)} = \|\mu\|_{\mathcal{M}_b(\Omega)} = \|\mu^+\|_{\mathcal{M}_b(\Omega)} + \|\mu^-\|_{\mathcal{M}_b(\Omega)} \end{aligned}$$

we obtain easily the first relation in (2.5). The second one is obtained in a similar way. \square

In order to proceed we need to prove that when both the classical formulation for the obstacle problem and the new one, given in Definition 1.3, make sense then the solutions, when they exist, are the same. At present we prove it for a nonpositive obstacle, and we will prove it in the general case in Section 5.

Lemma 2.4. *Let μ be an element of $\mathcal{M}_b(\Omega) \cap H^{-1}(\Omega)$ and ψ a nonpositive function; then the solution of $VI(\mu, \psi)$ coincides with the solution of $OP(\mu, \psi)$.*

Proof. Let u be the solution of $VI(\mu, \psi)$ and λ be the corresponding obstacle reaction. Thanks to Theorem 2.2 it is an element of $\mathcal{M}_b(\Omega)$; so $u \in \mathcal{F}_\psi(\mu)$. Take v an element in $\mathcal{F}_\psi(\mu)$, then $v = u_\mu + u_\nu$, with $\nu \in \mathcal{M}_b^+(\Omega)$, and $v \geq \psi$ q.e. in Ω .

Consider the approximation of ν , given by $\mathcal{A}T_k(u_\nu) =: \nu_k$. This is such that $\nu_k \rightharpoonup \nu$ $*$ -weakly in $\mathcal{M}_b(\Omega)$ and $\nu_k \in \mathcal{M}_b^+(\Omega) \cap H^{-1}(\Omega)$ (see [5]). Set $v_k = u_\mu + u_{\nu_k} = u_\mu + T_k(u_\nu)$. Since trivially $T_k(u_\nu) \nearrow u_\nu$ q.e. in Ω , we have

$$v_k \nearrow v \text{ q.e. in } \Omega.$$

Denote now the solutions of $VI(\mu, \psi_k)$ by u_k , where ψ_k are the functions defined by

$$\psi_k := \psi \wedge v_k.$$

From $\psi_k \leq \psi_{k+1}$ q.e. in Ω it easily follows that $u_k \leq u_{k+1}$ q.e. in Ω . Then there exists a function u^* such that $u_k \nearrow u^*$ q.e. in Ω .

So, passing to the limit in $u_k \geq \psi_k$ q.e. in Ω we obtain $u^* \geq \psi$ q.e. in Ω .

Moreover it is easy to see that $\|u_k\|_{H_0^1(\Omega)} \leq C$. So, thanks to Lemma 1.2 in [9] we get that u^* is a quasi continuous function of $H_0^1(\Omega)$ such that

$$u_k \rightharpoonup u^* \text{ weakly in } H_0^1(\Omega).$$

Moreover it can be easily proved that u^* is the solution of $VI(\mu, \psi)$ and by uniqueness $u^* = u$ q.e. in Ω .

Naturally, from the minimality of u_k , we deduce

$$u_k \leq v_k \text{ q.e. in } \Omega.$$

so, passing to the limit as $k \rightarrow +\infty$ we conclude that $u \leq v$ q.e. in Ω . Since this is true for every $v \in \mathcal{F}_\psi(\mu)$, the function u is the minimum in $\mathcal{F}_\psi(\mu)$, i.e. the solution of $OP(\mu, \psi)$. \square

We are now in position to prove that, for every $\mu \in \mathcal{M}_b(\Omega)$ and for every $\psi \leq 0$, there exists a solution to the Obstacle Problem according to Definition 1.3.

Theorem 2.5. *Let $\psi \leq 0$ and $\mu \in \mathcal{M}_b(\Omega)$. Then there exists a solution of $OP(\mu, \psi)$.*

Proof. Consider the function u_μ and define

$$\mathcal{A}(T_k(u_\mu)) =: \mu_k.$$

We know from [5] that

$$\mu_k \rightharpoonup \mu \quad \text{*}-\text{weakly in } \mathcal{M}_b(\Omega)$$

and $\mu_k \in H^{-1}(\Omega)$.

Let u_k be the solution of $VI(\mu_k, \psi)$ and denote

$$\mathcal{A}u_k - \mu_k =: \lambda_k,$$

which we know from Theorem 2.2 to be a measure in $\mathcal{M}_b^+(\Omega)$ such that

$$\|\lambda_k\|_{\mathcal{M}_b(\Omega)} \leq \|\mu_k^-\|_{\mathcal{M}_b(\Omega)}. \quad (2.6)$$

Up to a subsequence $\lambda_k \rightharpoonup \lambda$ *-weakly in $\mathcal{M}_b(\Omega)$, $u_k \rightarrow u$ strongly in $W_0^{1,q}(\Omega)$, with $u = u_\mu + u_\lambda$, and also $T_h(u_k) \rightharpoonup T_h(u)$ weakly in $H_0^1(\Omega)$, for all $h > 0$.

Now the set

$$E := \{v \in H_0^1(\Omega) : v \geq T_h(\psi) \text{ q.e. in } \Omega\}$$

is closed and convex in $H_0^1(\Omega)$, so it is also weakly closed. Since, clearly, $T_h(u_k) \geq T_h(\psi)$ q.e. in Ω , passing to the limit as $k \rightarrow +\infty$ we get that also $T_h(u) \in E$, hence $T_h(u) \geq T_h(\psi)$ q.e. in Ω for all $h > 0$. Passing to the limit as $h \rightarrow +\infty$ we get $u \geq \psi$ q.e. in Ω . In conclusion we deduce $u \in \mathcal{F}_\psi(\mu)$.

To show that u is minimal, take $v \in \mathcal{F}_\psi(\mu)$ so that $v \geq \psi$ and $v = u_\mu + u_\nu$.

Let $v_k = u_{\mu_k} + u_\nu$ so that $v_k = T_k(u_\mu) + u_\nu$ and $v_k \rightarrow v$ strongly in $W_0^{1,q}(\Omega)$.

Since $\psi \leq 0$, we have that $v_k \geq \psi$ q.e. in Ω . As u_k is the minimum of $\mathcal{F}_\psi(\mu_k)$, by Lemma 2.4, we obtain $u_k \leq v_k$ a.e. in Ω and in the limit $u \leq v$ a.e. in Ω . Hence u solves $OP(\mu, \psi)$. \square

From formula (2.6) we see that to extend (2.4) to the case of $\mu \in \mathcal{M}_b(\Omega)$ we just need to show that

$$\|\mu_k^-\|_{\mathcal{M}_b(\Omega)} \rightarrow \|\mu^-\|_{\mathcal{M}_b(\Omega)};$$

this is proved in the following Proposition.

Proposition 2.6. *Let $\psi \leq 0$ and $\mu \in \mathcal{M}_b(\Omega)$. Let u be the solution of $OP(\mu, \psi)$ and λ the corresponding obstacle reaction. Then*

$$\|\lambda\|_{\mathcal{M}_b(\Omega)} \leq \|\mu^-\|_{\mathcal{M}_b(\Omega)}.$$

Proof. What we need is implicit in [5]; we recall the main steps of that proof, having a closer look to the constants involved.

Let f_n be a smooth approximation of μ in the $*$ -weak topology of $\mathcal{M}_b(\Omega)$, such that $\|f_n\|_{L^1(\Omega)} \leq \|\mu\|_{\mathcal{M}_b(\Omega)}$, and let u_n be the solutions of

$$\begin{cases} \mathcal{A}u_n = f_n & \text{in } H^{-1}(\Omega) \\ u_n \in H_0^1(\Omega). \end{cases}$$

Consider, for $\delta > 0$, the Lipschitz continuous functions h_δ defined by

$$\begin{cases} h_\delta(s) = 1 & \text{if } |s| \leq k \\ h_\delta(s) = 0 & \text{if } |s| \geq k + \delta \\ |h'_\delta(s)| = \frac{1}{\delta} & \text{if } k \leq |s| \leq k + \delta, \end{cases}$$

and S_δ defined by

$$\begin{cases} S_\delta(s) = 0 & \text{if } |s| \leq k \\ S_\delta(s) = \text{sign}(s) & \text{if } |s| \geq k + \delta \\ S'_\delta(s) = \frac{1}{\delta} & \text{if } k \leq |s| \leq k + \delta. \end{cases}$$

Using the equation, we can see that $-\text{div}(h_\delta(u_n)A(x)\nabla u_n)$ belongs to $L^1(\Omega)$ and that

$$\begin{aligned} & \int_{\Omega} |-\text{div}(h_\delta(u_n)A(x)\nabla u_n)| \, dx \\ & \leq \int_{\Omega} |f_n| (h_\delta(u_n) + S_\delta^+(u_n) + S_\delta^-(u_n)) \, dx \\ & = \int_{\Omega} |f_n| \, dx \leq \|\mu\|_{\mathcal{M}_b(\Omega)}. \end{aligned}$$

This implies

$$\|\mu_k\|_{\mathcal{M}_b(\Omega)} \leq \|\mu\|_{\mathcal{M}_b(\Omega)},$$

(recall that $\mu_k = \mathcal{A}T_k(u_\mu)$) and we conclude thanks to Lemma 2.3. \square

3. The general existence theorem

We come now to prove the existence and uniqueness of the solution to the Obstacle Problem, without the technical assumption that the obstacle be negative. From now on the only hypothesis will be (1.4).

Theorem 3.1. *Let ψ satisfy (1.4) and let $\mu \in \mathcal{M}_b(\Omega)$. Then there exists a (unique) solution of $OP(\mu, \psi)$.*

Proof. It is enough to show that we can refer to the case $\psi \leq 0$. Indeed define

$$\varphi := \psi - u_\rho,$$

which is, obviously, negative.

By Theorem 2.5 there exists v minimum in $\mathcal{F}_\varphi(\mu - \rho)$, and we prove that the function $u := v + u_\rho$ is the minimum of $\mathcal{F}_\psi(\mu)$.

Trivially $u \geq \psi$ and, denoted the positive obstacle reaction associated to v by λ , we have $u = v + u_\rho = u_\mu + u_\lambda$, which shows that u is an element of $\mathcal{F}_\psi(\mu)$.

Consider now a function $w \in \mathcal{F}_\psi(\mu)$. By similar computations we deduce that $w - u_\rho$ belongs to $\mathcal{F}_\varphi(\mu - \rho)$ and, by the minimality of v , $v \leq w - u_\rho$, so that we conclude $u \leq w$ q.e. in Ω , and λ is the obstacle reaction associated to u . \square

Remark 3.2. From the previous proof we deduce that in the general case we have the inequality

$$\|\lambda\|_{\mathcal{M}_b(\Omega)} \leq \|(\mu - \rho)^-\|_{\mathcal{M}_b(\Omega)}. \quad (3.1)$$

4. Some stability results

In this section we want to show some results of continuous dependence of the solutions on the data.

The following proposition concerns the problem of stability with respect to the obstacle, which, however, is not true in general (see Remark 7.2).

Proposition 4.1. *Let $\psi_n : \Omega \rightarrow \overline{\mathbb{R}}$ be obstacles such that*

$$\psi_n \leq \psi \quad \text{and} \quad \psi_n \rightarrow \psi \quad \text{q.e. in } \Omega,$$

ψ satisfies (1.4), and let u_n and u be the solutions of $OP(\mu, \psi_n)$ and $OP(\mu, \psi)$, respectively. Then

$$u_n \rightarrow u \quad \text{strongly in } W_0^{1,q}(\Omega).$$

We also obtain that $u_n \rightarrow u$ q.e. in Ω and that $T_k(u_n) \rightharpoonup T_k(u)$ weakly in $H_0^1(\Omega)$, for all $k > 0$.

Proof. Since u is trivially in $\mathcal{F}_{\psi_n}(\mu)$ for any n we have

$$u_n \leq u \quad \text{q.e. in } \Omega. \tag{4.1}$$

To every minimum u_n there corresponds a positive obstacle reaction λ_n , satisfying inequality (3.1), so we obtain that, up to a subsequence,

$$\lambda_n \rightharpoonup \hat{\lambda} \quad \text{*weakly in } \mathcal{M}_b(\Omega)$$

$$u_n \rightarrow \hat{u} \quad \text{strongly in } W_0^{1,q}(\Omega)$$

and

$$\hat{u} = u_\mu + u_{\hat{\lambda}}.$$

Hence, from (4.1), $\hat{u} \leq u$ a.e. in Ω , and also q.e. On the other side, we have to prove that $\hat{u} \geq \psi$ q.e. in Ω , in order to obtain $\hat{u} \in \mathcal{F}_\psi(\mu)$, and so $u \leq \hat{u}$ q.e. in Ω .

First consider the case when $\psi_n \leq \psi_{n+1}$ q.e. in Ω .

From this fact it follows that $u_n \leq u_{n+1}$ q.e. in Ω , and then $T_k(u_n) \leq T_k(u_{n+1})$ q.e. in Ω , for all $k > 0$. Hence this sequence has a quasi everywhere limit. On the other hand, the fact that $\mu + \lambda_n \rightharpoonup \mu + \hat{\lambda}$ *-weakly in $\mathcal{M}_b(\Omega)$ implies that $T_k(u_n) \rightharpoonup T_k(\hat{u})$ weakly in $H_0^1(\Omega)$ and then, by Lemma 1.2 of [9], $T_k(u_n) \rightarrow T_k(\hat{u})$ q.e. in Ω . Since this holds for all $k > 0$ we get also

$$u_n \rightarrow \hat{u} \quad \text{q.e. in } \Omega.$$

Then, passing to the limit in $u_n \geq \psi_n$ q.e. in Ω we get $\hat{u} \geq \psi$ q.e. in Ω .

If the sequence ψ_n is not increasing, consider

$$\varphi_n := \inf_{k \geq n} \psi_k, \tag{4.2}$$

so that $\varphi_n \nearrow \psi$ q.e. in Ω and $\varphi_n \leq \psi_n$ q.e. in Ω . If \bar{u}_n is the solution of $OP(\mu, \varphi_n)$ it is easy to see, using Definition 1.3, that $\bar{u}_n \leq u_n \leq u$ q.e. in Ω . Applying the first case to \bar{u}_n and passing to the limit we get $u_n \rightarrow u$ q.e. in Ω . \square

As for stability with respect to the right-hand side, we will show later that in general it is not true that if

$$\mu_n \rightharpoonup \mu \quad \text{*}-\text{weakly in } \mathcal{M}_b(\Omega)$$

then

$$u_n \rightarrow u \text{ strongly in } W_0^{1,q}(\Omega),$$

where u_n and u are the solutions relative to μ_n and μ with the fixed obstacle ψ .

However we can give now the following stability result.

Proposition 4.2. *Let μ_n and μ be measures in $\mathcal{M}_b(\Omega)$ such that*

$$\mu_n \rightarrow \mu \text{ strongly in } \mathcal{M}_b(\Omega),$$

then

$$u_n \rightarrow u \text{ strongly in } W_0^{1,q}(\Omega)$$

where u_n and u are the solutions of $OP(\mu_n, \psi)$ and of $OP(\mu, \psi)$, respectively.

Proof. Let λ_n be the obstacle reactions associated to u_n , then

$$\|\lambda_n\|_{\mathcal{M}_b(\Omega)} \leq \|(\mu_n - \rho)^-\|_{\mathcal{M}_b(\Omega)},$$

so, up to a subsequence,

$$\lambda_n \rightharpoonup \hat{\lambda} \quad \text{*}-\text{weakly in } \mathcal{M}_b(\Omega)$$

and

$$u_n \rightarrow \hat{u} \text{ strongly in } W_0^{1,q}(\Omega)$$

$$T_k(u_n) \rightharpoonup T_k(\hat{u}) \text{ weakly in } H_0^1(\Omega) \quad \forall k > 0$$

where $\hat{u} = u_\mu + u_{\hat{\lambda}}$.

As $T_k(u_n) \geq T_k(\psi)$ q.e. in Ω for every $k \geq 0$, and for every n , we have $T_k(\hat{u}) \geq T_k(\psi)$ q.e. in Ω for every $k > 0$.

Passing to the limit as $k \rightarrow +\infty$ we obtain that \hat{u} belongs to $\mathcal{F}_\psi(\mu)$.

Let $v \in \mathcal{F}_\psi(\mu)$, with ν the associated measure. Consider now v_n the Stampacchia solution relative to $\zeta_n := \mu_n + (\mu_n - \mu)^- + \nu$. Since $\zeta_n \rightarrow \mu + \nu$ strongly in $\mathcal{M}_b(\Omega)$, the sequence v_n converges strongly in $W_0^{1,q}(\Omega)$ to v .

Moreover $v_n \geq v \geq \psi$ q.e. in Ω ; hence $v_n \in \mathcal{F}_\psi(\mu_n)$, then $u_n \leq v_n$ q.e. in Ω , and, in the limit,

$$\hat{u} \leq v \quad \text{a.e. in } \Omega,$$

and hence also q.e. in Ω .

□

Remark 4.3. Thanks to this last result we can say that the solutions obtained in this paper coincide with those given by Boccardo and Cirmi in [1] and [2] when the data are $L^1(\Omega)$ -functions.

As said above we give now the counterexample showing that in general there is not stability with respect to $*$ -weakly convergent data.

Example 4.4 Let $\Omega = (0, 1)^N$ with $N \geq 3$, $\mathcal{A} = -\Delta$ and $\psi \equiv 0$.

The construction follows the one made by Cioranescu and Murat in [7].

For each $n \in \mathbb{N}$, divide the whole of Ω into small cubes of side $\frac{1}{n}$. In the centre of each of them take two balls: $B_{\frac{1}{2n}}$, inscribed in the cube, and B_{r_n} of ray $r_n = \left(\frac{1}{2n}\right)^{\frac{N}{N-2}}$.

In each cube define w_n to be the capacitary potential of B_{r_n} with respect to $B_{\frac{1}{2n}}$ extended by zero in the rest of the cube.

Hence

$$\Delta w_n = \mu_n,$$

with

$$\mu_n \rightharpoonup 0 \text{ both weakly in } H^{-1}(\Omega) \text{ and } * \text{-weakly in } \mathcal{M}_b(\Omega).$$

(see [7]). Thus $w_n \rightharpoonup 0$ weakly in $H_0^1(\Omega)$.

Let u_n be the solution of $VI(\mu_n, 0)$. Using w_n as test function in the Variational Inequality we get $\|u_n\|_{H_0^1(\Omega)} \leq C$. By contradiction assume that its $H_0^1(\Omega)$ -weak limit is zero.

Consider the function $z_n := u_n + w_n$ which must then converge to zero weakly in $H_0^1(\Omega)$. Obviously $z_n \geq w_n$ q.e. in Ω and then $z_n \geq 1$ on $\bigcup B_{r_n}$. Hence if we define the obstacles

$$\psi_n := \begin{cases} 1 & \text{in } \bigcup B_{r_n} \\ 0 & \text{elsewhere} \end{cases}$$

$z_n \geq \psi_n$. Call v_n the function realizing

$$\min_{\substack{v \geq \psi_n \\ v \in H_0^1(\Omega)}} \int_{\Omega} |\nabla v|^2 dx.$$

A simple computation yields

$$-\Delta z_n = -\Delta u_n - \Delta w_n \geq 0.$$

Then $z_n \geq v_n \geq 0$, so that

$$v_n \rightharpoonup 0 \text{ weakly in } H_0^1(\Omega).$$

But this is not possible because a Γ -convergence result contained in [6] says that there exists a constant $c > 0$ such that v_n tends to the minimum point of

$$\min_{\substack{v \geq 0 \\ v \in H_0^1(\Omega)}} \int_{\Omega} |\nabla v|^2 dx + c \int_{\Omega} |(v-1)^-|^2 dx$$

which is not zero.

5. Comparison with the classical solutions

As announced, in this section, we want to show that the new formulation of Obstacle Problem is consistent with the classical one.

To talk about the equivalence of the two formulations it is necessary that both make sense. So we will work under the hypothesis that $\mu \in \mathcal{M}_b(\Omega) \cap H^{-1}(\Omega)$ and that the obstacle ψ satisfies

$$\exists z \in H_0^1(\Omega) \text{ s.t. } z \geq \psi \text{ q.e. in } \Omega; \quad (5.1)$$

$$\exists \rho \in \mathcal{M}_b^+(\Omega) \text{ s.t. } u_\rho \geq \psi \text{ q.e. in } \Omega. \quad (5.2)$$

Later on we will discuss these conditions in deeper details.

Lemma 5.1. *If there exists a measure $\sigma \in \mathcal{M}_b(\Omega) \cap H^{-1}(\Omega)$ such that $u_\sigma \geq \psi$ q.e. in Ω , then the solutions of $VI(\mu, \psi)$ and of $OP(\mu, \psi)$ coincide.*

Proof. Let u be the solution of $VI(\mu, \psi)$. Subtracting u_σ to it, and with the same technique as in the proof of Theorem 3.1, we return to the case of negative obstacle and we can use Lemma 2.4. \square

Theorem 5.2. *Under the hypotheses (5.1) and (5.2), the solutions of $VI(\mu, \psi)$ and of $OP(\mu, \psi)$ coincide.*

Proof. As a first step consider the case of an obstacle bounded from above by a constant M . The measure $\rho_M := \mathcal{A}(T_M(u_\rho))$ is in $\mathcal{M}_b(\Omega) \cap H^{-1}(\Omega)$ and $T_M(u_\rho) \geq \psi$ so that we are in the hypotheses of the previous lemma.

If, instead, ψ is not bounded, we consider $\psi \wedge k$, and, with respect to this new obstacle, conditions (5.1) and (5.2) are satisfied by the function $T_k(u_\rho)$.

Hence we can apply the first step and say that u_k , solution of $VI(\mu, T_k(\psi))$, is also the solution of $OP(\mu, T_k(\psi))$.

From the classical theory we know that the sequence u_k tends in $H_0^1(\Omega)$ to the solution of $VI(\mu, \psi)$, while from Proposition 4.1 u_k converges in $W_0^{1,q}(\Omega)$ to the solution of $OP(\mu, \psi)$. \square

A little attention is required in treating conditions (5.1) and (5.2). Each one is necessary for the corresponding problem to be nonempty, but together they can be somewhat weakened.

First of all we underline that no one of the two conditions is implied by the other. This is seen with the following examples.

Example 5.3. Let $\Omega = (-1, 1) \subset \mathbb{R}$ and let $\mathcal{A} = -\Delta = -u''$. Take $\psi \in H_0^1(-1, 1)$ such that $-\psi''$ is an unbounded positive Radon measure. For instance we may take $\psi = (1 - |x|)(1 - \log(1 - |x|))$.

Now (5.1) is trivially true, and the solution of $VI(0, \psi)$ is ψ itself. If also (5.2) were true, then ψ would be also the solution of $OP(0, \psi)$. But this is not possible, because, being $-\psi''$ an unbounded measure, we can not write it as u_λ for some $\lambda \in \mathcal{M}_b^+(\Omega)$.

Example 5.4. Let $N \geq 3$, $\mathcal{A} = -\Delta$ and $\rho = \delta_{x_0}$, the Dirac delta in a fixed point $x_0 \in \Omega$.

Take $\psi = u_{\delta_{x_0}}$, the Green function with pole at x_0 . Then (5.2) holds, but if also (5.1) held we would have $\psi \in L^{2^*}(\Omega)$ which is not true.

On the other side we already saw in the proof of Theorem 5.2 that if we add to Condition (5.2) the assumption that the obstacle be bounded, this is enough for (5.1) too to hold.

Moreover, if, besides (5.1), we assume that the obstacle is “controlled near the boundary” also Condition (5.2) is true:

Assume that (5.1) holds and there exists a compact $J \subset \Omega$, such that $\psi \leq 0$ in $\Omega \setminus J$. Then also (5.2) holds. Indeed just take as ρ the obstacle reaction corresponding to u , the solution of $VI(0, \psi)$. Then

$$\text{supp} \rho \subset J,$$

and $\rho \in \mathcal{M}_b^+(\Omega)$.

A finer condition expressing the “control near the boundary” is

(2') $\exists J$ compact $\subset \Omega$ and $\exists \tau \in \mathcal{M}_b^+(\Omega) \cap H^{-1}(\Omega) : u_\tau \geq \psi$ in $\Omega \setminus J$.

In conclusion we want to remark that, in general, in classical Variational Inequalities, the obstacle reaction associated to the solution is indeed a Radon measure, but it is not always bounded, as Example 5.3 shows.

On the other side, in the new setting, the minimum of $\mathcal{F}_\psi(\mu)$ is not, in general, an element of $H_0^1(\Omega)$.

Hence the two formulations do not overlap completely and no one is included in the other.

6. Approximation properties

As we have seen so far, if we have a sequence μ_n $*$ -weakly convergent to μ , we can not deduce convergence of solutions, but, from (3.1) we have

$$\|\lambda_n\|_{\mathcal{M}_b(\Omega)} \leq \|(\mu - \rho)^-\|_{\mathcal{M}_b(\Omega)},$$

where the λ_n are the obstacle reactions relative to the solutions u_n . So, up to a subsequence,

$$\lambda_n \rightharpoonup \hat{\lambda} \quad * \text{-weakly in } \mathcal{M}_b(\Omega)$$

and

$$u_n \rightarrow \hat{u} = u_\mu + u_{\hat{\lambda}} \text{ strongly in } W_0^{1,q}(\Omega).$$

With the same argument used in the proof of Theorem 2.5 we can show that $\hat{u} \geq \psi$ q.e. in Ω . Hence $\hat{u} \geq u$, the minimum of $\mathcal{F}_\psi(\mu)$.

On the other hand, in Theorem 3.1 we have obtained the solution of $OP(\mu, \psi)$ as a limit of the solutions to $OP(AT_n(u_{\mu-\rho}) + \rho, \psi)$. We remark that if ρ belongs to the

ordered dual of $H_0^1(\Omega)$ that is $V := \{\mu \in \mathcal{M}_b(\Omega) \cap H^{-1}(\Omega) : |\mu| \in H^{-1}(\Omega)\}$, then the approximating problems are actually Variational Inequalities.

Thanks to these two facts we can characterize the solution u of $OP(\mu, \psi)$ by approximation with solutions of Variational Inequalities with data in V as follows.

1. For every sequence μ_n in $\mathcal{M}_b(\Omega)$, with $\mu_n \rightharpoonup \mu$ $*$ -weakly in $\mathcal{M}_b(\Omega)$, we have

$$s\text{-}W_0^{1,q}(\Omega)\text{-}\lim_{n \rightarrow \infty} u_n \geq u.$$

2. There exists a sequence $\mu_n \in V$, with $\mu_n \rightharpoonup \mu$ $*$ -weakly in $\mathcal{M}_b(\Omega)$ such that

$$s\text{-}W_0^{1,q}(\Omega)\text{-}\lim_{n \rightarrow \infty} u_n = u$$

In other words:

$$u = \min \left\{ s\text{-}\lim_{n \rightarrow +\infty} u_n : u_n \text{ sol. } VI(\mu_n, \psi), \mu_n \in V, \mu_n \rightharpoonup \mu \text{ } * \text{-weakly in } \mathcal{M}_b(\Omega) \right\}.$$

7. Measures vanishing on sets of zero Capacity

We show now an example (suggested by L. Orsina and A. Prignet) in which the solution of the Obstacle Problem with right-hand side measure does not touch the obstacle, though it is not the solution of the equation.

Example 7.1. Let $N \geq 2$, Ω be the ball $B_1(0)$, and $\mathcal{A} = -\Delta$. Take the datum μ a negative measure concentrated on a set of zero 2-Capacity and the obstacle ψ negative and bounded below by a constant $-h$. Let u be the solution of $OP(\mu, \psi)$, then $u = u_\mu + u_\lambda$. We want to show that $\lambda = -\mu$.

First observe that, for minimality, $u \leq 0$; on the other hand $u \geq -h$, so that $u = T_h(u)$ and hence $u \in H_0^1(\Omega)$. This implies that the measure $\mu + \lambda$ is in $\mathcal{M}_b(\Omega) \cap H^{-1}(\Omega)$, which is contained in $\mathcal{M}_b^0(\Omega)$, the measures which are zero on the sets of zero 2-Capacity (see [4]). In other words $\lambda = -\mu + \hat{\lambda}$, with $\hat{\lambda}$ a measure in $\mathcal{M}_b^0(\Omega)$, and so positive, since λ is positive. Then $u \geq 0$, and finally $u = 0$. Thus the solution can be far above the obstacle, but the obstacle reaction is nonzero, and is exactly $-\mu$.

Remark 7.2. This example shows also that in general there is no continuous dependence on the obstacles. Indeed, if $h \rightarrow +\infty$, then the solution of $OP(\mu, -h)$ is identically zero for each h , while the solution of $OP(\mu, -\infty)$ is u_μ .

We want to consider here a class of data for which the above phenomenon is avoided.

Consider, as datum, a measure in $\mathcal{M}_b^0(\Omega)$. In this case we can use the fact (contained in [4]) that for any such measure μ there exists a function f in $L^1(\Omega)$ and a functional F in $\mathcal{M}_b(\Omega) \cap H^{-1}(\Omega)$, such that $\mu = f + F$. If, in addition $\mu \geq 0$, then also f can be taken to be positive.

We want to show that also the obstacle reaction λ belongs to $\mathcal{M}_b^0(\Omega)$ and that in this particular case we can write our Obstacle Problem in a variational way, that is with and “entropy formulation”.

We begin by considering the case of a negative obstacle.

Lemma 7.3. *Let $\psi \leq 0$ and let $\mu_1, \mu_2 \in \mathcal{M}_b(\Omega) \cap H^{-1}(\Omega)$. Let λ_1 and λ_2 be the reactions of the obstacle corresponding to the solutions u_1 and u_2 of $VI(\mu_1, \psi)$ and $VI(\mu_2, \psi)$, respectively.*

If $\mu_1 \leq \mu_2$ then $\lambda_1 \geq \lambda_2$.

Proof. This proof is inspired by Lemma 2.5 in [11]. We easily have that $u_1 \leq u_2$.

Take now a function $\varphi \in \mathcal{D}(\Omega)$, $\varphi \geq 0$, and set

$$\varphi_\varepsilon := \varepsilon \varphi \wedge (u_2 - u_1) \in H_0^1(\Omega).$$

Now, using the hypothesis that $\mu_1 \leq \mu_2$ and monotonicity of \mathcal{A} , compute

$$\begin{aligned} \langle \lambda_1, \varepsilon \varphi - \varphi_\varepsilon \rangle &\geq \langle \mathcal{A}u_1, \varepsilon \varphi - \varphi_\varepsilon \rangle - \langle \mu_2, \varepsilon \varphi - \varphi_\varepsilon \rangle \\ &= \langle \mathcal{A}u_1 - \mathcal{A}u_2, \varepsilon \varphi - \varphi_\varepsilon \rangle + \langle \lambda_2, \varepsilon \varphi - \varphi_\varepsilon \rangle \\ &\geq \varepsilon \int_{\{u_2 - u_1 \leq \varepsilon \varphi\}} A(x) \nabla(u_1 - u_2) \nabla \varphi + \varepsilon \langle \lambda_2, \varphi \rangle - \langle \lambda_2, \varphi_\varepsilon \rangle. \end{aligned}$$

Now, using u_1 as a test function in $VI(\mu_2, \psi)$ and the fact that $u_2 - u_1 \geq \varphi_\varepsilon \geq 0$ we easily get $\langle \lambda_2, \varphi_\varepsilon \rangle = 0$.

Since, also, $-\langle \lambda_1, \varphi_\varepsilon \rangle \leq 0$ we obtain

$$\langle \lambda_1, \varphi \rangle \geq \int_{\{u_2 - u_1 \leq \varepsilon \varphi\}} A(x) \nabla(u_1 - u_2) \nabla \varphi + \langle \lambda_2, \varphi \rangle.$$

Passing to the limit as $\varepsilon \rightarrow 0$ and observing that

$$\int_{\{u_2 - u_1 \leq \varepsilon \varphi\}} A(x) \nabla(u_1 - u_2) \nabla \varphi \longrightarrow \int_{\{u_2 = u_1\}} A(x) \nabla(u_1 - u_2) \nabla \varphi = 0,$$

we get the thesis. \square

Let us see now what can we say more if $\mu \in \mathcal{M}_b^0(\Omega)$, still in the case of negative obstacle.

Lemma 7.4. *Let $\psi \leq 0$ and let $\mu \in \mathcal{M}_b^0(\Omega)$ then the obstacle reaction relative to the solution of $OP(\mu, \psi)$ is also in $\mathcal{M}_b^0(\Omega)$.*

Proof. It is not restrictive to assume μ to be negative. Indeed, if $\mu = \mu^+ - \mu^-$, then also μ^+ and μ^- are in $\mathcal{M}_b^0(\Omega)$. Hence the minimum of $\mathcal{F}_\psi(\mu)$ can be written as $u_{\mu^+} + v$ with v minimum in $\mathcal{F}_{\psi - u_{\mu^+}}(-\mu^-)$, and the same obstacle reaction λ ; and so we are in the case of a negative measure.

Consider now the decomposition $\mu = f + F$ with $f \leq 0$. And let $\mu_k := T_k(f) + F$ so that $\mu_k \rightarrow \mu$ strongly in $\mathcal{M}_b(\Omega)$.

Let u_k be the solution of $OP(\mu_k, \psi)$. It is also the solution of $VI(\mu_k, \psi)$ so that $\lambda_k \in \mathcal{M}_b^0(\Omega)$.

Thanks to Proposition 4.2 we have that $u_k \rightarrow u = u_\mu + u_\lambda$ strongly in $W_0^{1,q}(\Omega)$ and that $\lambda_k \rightharpoonup \lambda$ *-weakly in $\mathcal{M}_b(\Omega)$.

From the fact that $\mu_k \geq \mu_{k+1}$ and from Lemma 7.3 we obtain that $\lambda_k \leq \lambda_{k+1}$. Hence if we define

$$\hat{\lambda}(B) := \lim_{k \rightarrow \infty} \lambda_k(B) \quad \forall B \text{ Borel set in } \Omega,$$

we know from classical measure theory that it is a bounded Radon measure, it is in $\mathcal{M}_b^0(\Omega)$, since all λ_k are, and necessarily coincides with λ . So $\lambda \in \mathcal{M}_b^0(\Omega)$. \square

In order to pass to a signed obstacle observe first that the minimal hypothesis (1.4) becomes necessarily

$$\exists \sigma \in \mathcal{M}_b^0(\Omega) : u_\sigma \geq \psi. \quad (7.1)$$

Once we have noticed this, it is easy to use the result for a negative obstacle, as we did in the proof of Theorem 3.1 and obtain the following result.

Theorem 7.5. *Let ψ satisfy hypothesis (7.1), and let μ be in $\mathcal{M}_b^0(\Omega)$. Then the obstacle reaction relative to the solution of $OP(\mu, \psi)$ belongs to $\mathcal{M}_b^0(\Omega)$ as well.*

Remark 7.6. Notice that thanks to the pointwise convergence we have, in this case, that $\lambda_k \rightarrow \lambda$ strongly in $\mathcal{M}_b(\Omega)$.

Remark 7.7. These properties of the case of $\mathcal{M}_b^0(\Omega)$ measures, allow us to write the Obstacle Problem in a “more variational” way. Namely, if $\mu \in \mathcal{M}_b^0(\Omega)$ and its decomposition is $\mu = f + F$ then the function u solution of $OP(\mu, \psi)$ satisfies also

$$\begin{cases} \langle \mathcal{A}u, T_j(v - u) \rangle \geq \int_{\Omega} f T_j(v - u) + \langle F, T_j(v - u) \rangle \\ \forall v \in H_0^1(\Omega) \cap L^\infty(\Omega), v \geq \psi \text{ q.e. in } \Omega \end{cases}.$$

This is similar to the entropy formulation given by Boccardo and Cirmi in [1] in the case of datum in $L^1(\Omega)$. The proof that such a formulation holds is made by approximation. To this aim we choose a particular sequence of measures $\mu_k := T_k(f) + F$, so that $\mu_k \rightarrow \mu$ strongly in $\mathcal{M}_b(\Omega)$. Hence also the solutions of $OP(\mu_k, \psi)$ (and also of $VI(\mu_k, \psi)$) converge strongly in $W_0^{1,q}(\Omega)$ to u solution of $OP(\mu, \psi)$. Then u_k solves

$$\begin{cases} \langle \mathcal{A}u_k, v - u_k \rangle \geq \langle \mu_k, v - u \rangle \\ \forall v \in H_0^1(\Omega), v \geq \psi \text{ q.e. in } \Omega \end{cases}$$

In this inequality we can use as test functions $v = T_j(w - u_k) + u_k$, with $w \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $w \geq \psi$ q.e. in Ω , and, by calculations similar to those in [1], get the result.

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